1101. Solve the equation $3x^{10} + 8x^6 = 0$.

1102. By multiplying out and equating coefficients, or otherwise, write the expression $12x^2 + 2x - 3$ in the form $a(2x + 1)^2 + b(2x + 1) + c$.

1103. Shade the region of the (x, y) plane which satisfies both of the following inequalities:

$$|x| \geq 1, \qquad |y| \leq 1.$$

1104. A pilot sample of ten data is taken from a large population. Find the probability that all ten data lie between the quartiles of the population.

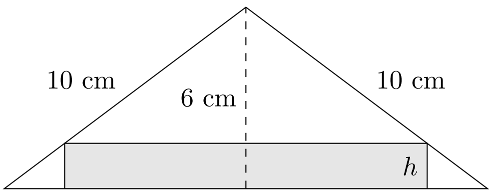
1105. Solve the equation $\dfrac{2x-1}{3x-2} = \dfrac{4x-3}{5x-4}$.

1106. A linear function g is defined from a function f, in terms of an unknown constant a, by

$$\mathrm{g}(x) = \mathrm{f}'(a)(x - a) + \mathrm{f}(a).$$

Show that $y = \mathrm{g}(x)$ is tangent to $y = \mathrm{f}(x)$.

1107. The diagram shows an isosceles triangle of height 6 cm and slant height 10 cm. A shaded rectangle is drawn inside the triangle, of variable height h.



(a) Show that the area of the rectangle is given by

$$A = 16h - \frac{8h^2}{3}.$$

(b) Hence, show that the maximum possible area of the rectangle is half that of the triangle.

1108. Simplify the following, where $n \in \mathbb{Z}$, giving your answers in standard form:

(a) $2.3 \times 10^n + 1.2 \times 10^{n-1}$,

(b) $5 \times 10^n + 9.7 \times 10^{n+1}$.

1109. Find the exact area of the region of the (x, y) plane defined by the simultaneous inequalities

$$x + y > 1,$$
$$x^2 + y^2 < 1.$$

1110. Simplify the following sets:

(a) $[0, 1] \cup [1, 2] \cup [2, 3]$,

(b) $[0, 3] \cap [1, 4] \cap [2, 5]$.

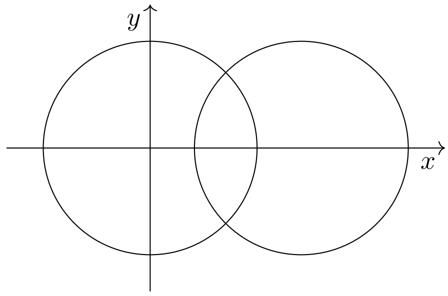
1111. For each of the following, state, with a reason, whether such an entity can exist:

(a) A function of the form $\mathrm{f}(x) = ax^2 + bx + c$, for constants $a, b, c \in \mathbb{R}$, which is linear.

(b) A prime expressible as mn, for $mn \in \mathbb{N}$.

1112. Write down the area scale factor when $y = \mathrm{f}(x)$ is transformed to $y = k\,\mathrm{f}(kx)$.

1113. The diagram shows the circles $(x - 2)^2 + y^2 = 2$ and $x^2 + y^2 = 2$:



Show that the circles are normal to one another.

1114. The function $\mathrm{f}(x) = x^n(x + 1)$, for $n \in \mathbb{N}$, has $\mathrm{f}''(x) = ax^3(bx + c)$, for $a, b, c \in \mathbb{N}$. Find n, a, b, c.

1115. Prove by contradiction that every pentagon must have at least one interior angle $\theta \geq 108°$.

1116. Electricity pylons stand every 50 metres alongside a railway track. A train accelerating constantly at a ms^{-2} covers successive gaps between pylons in 2.2 and 1.8 seconds.

(a) Show that initial speed u and a satisfy

$$50 = 2.2u + 2.42a,$$
$$100 = 4u + 8a.$$

(b) Hence, find u and a.

1117. Give the range of $\mathrm{g} : x \mapsto (\cos x + 3)^3$.

1118. A sample of forty bivariate data is being tested. The researcher is looking for evidence of negative correlation in the population. Defining ρ to be the population correlation coefficient, the researcher is using the following null hypothesis:

$$H_0 : \rho = 0.$$

(a) Write down the alternative hypothesis.

(b) The critical value for the test is $r_c = -0.264$, and the sample statistic is $r = -0.462$. Carry out the test.

1119. The graph $x = y^2$ is translated by the vector $a\mathbf{i} + b\mathbf{j}$. Write down the equation of the new graph.

1120. Write down the broadest real domains over which the following functions may be defined:

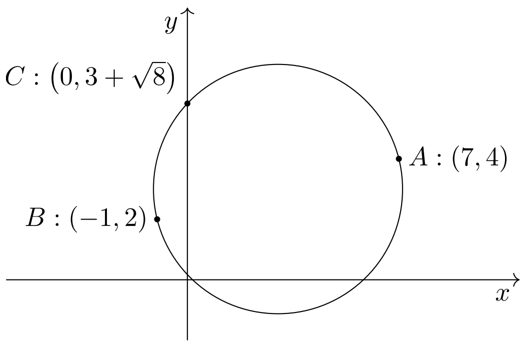
(a) $x \mapsto \sqrt{x-1}$,
(b) $x \mapsto \sqrt{1-x}$.

1121. Determine the exact roots of the equation

$$\sqrt{2}x^2 + \sqrt{24}x - \sqrt{8} = 0.$$

1122. When not loaded, a light washing line 4 m long is horizontal. A bag of clothes pegs weighing 0.8 kg is then hung in equilibrium at its centre. The centre of the line is now 20 cm below its initial position. Find the tension in the line.

1123. AB and CD are two diameters of the same circle. Points A, B, C are as shown:



Find the exact coordinates of D.

1124. The interior angles of a quadrilateral are in AP. Prove that the quadrilateral is convex, i.e that all of its interior angles are less than 180°.

1125. Show that $\displaystyle\int_1^8 \frac{40(x+1)^2}{\sqrt[3]{x}}\,dx = 5493$.

1126. Describe all functions f for which f'' is constant.

1127. A *gradian* is a decimal measure of angle, sometimes used in surveying and geology, in which a right angle contains 100 gradians. Determine formulae, with angle θ given in gradians, for arc length l and sector area A.

1128. Solve the equation $|x^2 - x| = |x - 1|$.

1129. Show that $\left(x-a+\sqrt{b}\right)\left(x-a-\sqrt{b}\right)$, where $a, b \in \mathbb{Z}$ and $b \geq 0$, is a quadratic with integer coefficients.

1130. A non-linear differential equation is given as

$$\mathrm{g}'(x) = 1 + \big(\mathrm{g}(x)\big)^2.$$

Verify that $\mathrm{g}(x) = \tan x$ satisfies this DE.

1131. It is given that there are constants A, B for which the following is an identity:

$$\frac{2x^2 + 3x + c}{2x - 1} \equiv Ax + B.$$

Find the value of the constant c.

1132. "The coordinate axes are normal to the curve $x^2 + 2x + y^2 = 1$." True or false?

1133. A bank account offers compound interest at 2% per annum, with the money paid into the account as a lump sum at the end of the year. Determine the number of years after which an initial investment gives a return of over 25%.

1134. If $u = 3x + 4$, write $x^2 + 6x + 1$ as a simplified quadratic in u.

1135. A sledge is sliding down a snowy slope. Explain how you know that the reaction exerted by the sledge on the slope and the friction exerted by the sledge on the slope are at right angles.

1136. Prove that, if $a, b \in \mathbb{Q}$, then $a - b \in \mathbb{Q}$.

1137. A function g is defined by $\mathrm{g} : x \mapsto x + |x| - 1$.

(a) Show that $\mathrm{g}(x) = -1$ for $x \leq 0$.
(b) Show that $\mathrm{g}(x) = 2x - 1$ for $x \geq 0$.
(c) Hence, sketch $y = \mathrm{g}(x)$.

1138. A function f is such that, for some constant k, the following equation holds for all x:

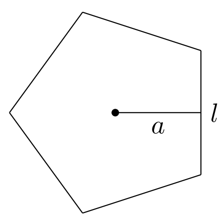
$$\int \mathrm{f}(x)\,dx = k\,\mathrm{f}(x) + c.$$

Prove that f cannot be a polynomial function.

1139. Find the length of the line segment

$$x = a + bt, \quad y = c + dt, \quad t \in \left[0, \frac{1}{\sqrt{b^2 + d^2}}\right].$$

1140. The *apothem* of a regular polygon is the distance from the centre to the midpoint of one of the sides.



Prove that, in an n-sided polygon, the apothem a is related to the side length l by the formula

$$a = \tfrac{1}{2} l \cot \tfrac{180°}{n}.$$

1141. Two APs have nth terms a_n and b_n. Prove that their mean $c_n = \frac{1}{2}(a_n + b_n)$ is also an AP.

1142. Parabolae $y = (x+1)(x-a)$ and $y = (x-2)^2 + b$ are mirror images in the line $x = 3$. Find a and b.

1143. An equation in y is given as

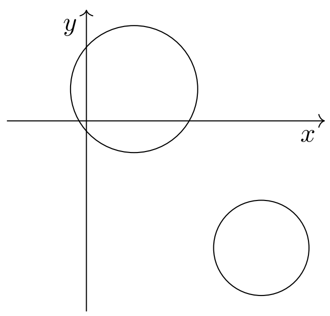
$$\Big[\log_x y\Big]_{x=2}^{x=4} = 3.$$

(a) Prove the general result $\log_{a^2} b \equiv \frac{1}{2}\log_a b$.

(b) Hence, solve for y.

1144. A car moves, for 10 seconds, with position given, in metres, by $x = t^2$. Find the time t during the motion at which the car's instantaneous speed is equal to its average speed over the ten seconds.

1145. The diagram shows circles C_1 and C_2:



The circles have equations

$$C_1 : (x-3)^2 + (y+2)^2 = 16,$$
$$C_2 : (x-11)^2 + (y+8)^2 = 9.$$

Find the shortest distance between C_1 and C_2.

1146. It is given that the functions $f(x) = -x^2 + kx + 4$ and $g(x) = -3x^2 - 6x + 5$ have the same range over $\mathbb{R}$. Find the value(s) of the constant k.

1147. Prove the following results:

(a) $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$,

(b) $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

1148. Using integration, show that the average value of the function $f(x) = x^2$ on the domain $[0, 12]$ is 48.

1149. Verify, by calculating derivatives, that the curve $y = \sqrt{x}$ satisfies the differential equation

$$4y^3\frac{d^2y}{dx^2} + 1 = 0.$$

1150. If $a = 2^x$ and $b = 3^x$, write a in terms of b.

1151. The points (a, a), $(0, a-1)$ and $(8, 5)$ are collinear. Find all possible values of a.

1152. State, with a reason, whether, in a game of cards, being dealt a straight (consecutive numbers) is more probable if the cards are picked

① with replacement,
② without replacement.

1153. Complete the square on $\sqrt{3}x^2 - \sqrt{48}x - \sqrt{27}$.

1154. Two concentric circles have the same centre, and radii 9 cm and 15 cm. A chord, which is tangent to the smaller circle, is drawn inside the larger circle. Determine the length of the chord.

1155. Show that the area of the region enclosed by the graphs $y = 3x^2$ and $y = |x| + 2$ is 3.

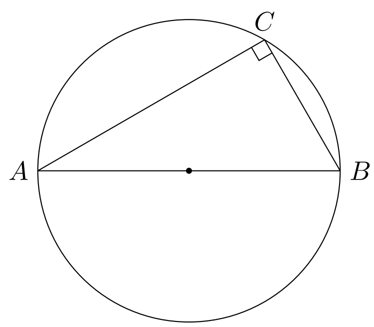
1156. A function is defined, over the domain $\mathbb{R}$, by the instruction $f : x \mapsto x^4 - x^2$.

(a) Find $f'(x)$ and $f''(x)$.

(b) Find all x values such that $f''(x) = 0$.

(c) Show, by considering the sign of $f''(x)$ either side of these x values, that these are points of inflection.

1157. Prove the "angle in a semicircle" theorem, which states that the angle subtended by a diameter at the circumference is a right angle.



1158. In each case, state whether the given events are independent:

(a) "Coin A shows heads"; "Coin A shows tails".

(b) "Coin A shows heads"; "Coin B shows tails".

1159. Solve ${}^n\mathrm{C}_3 - {}^n\mathrm{C}_2 = 0$, for $3 \le n \in \mathbb{N}$.

1160. Points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, relative to an origin O.

(a) Show that any point P on the line AB has position vector $\mathbf{p} = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$, for $\lambda \in \mathbb{R}$.

(b) Point C is now defined with position vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$. Find an expression, in terms of a new parameter μ, for the position vector $\mathbf{q}$ of any point Q along the line OC.

(c) Hence, or otherwise, prove that the diagonals of a parallelogram bisect each other.

1161. If $y = \sqrt{8x+7}$, find $\dfrac{dy}{dx}$.

1162. Triangles T_1 and T_2 have side lengths $(11, 60, 61)$ and $(11, l, 109)$. Their areas are the same.

(a) Show that T_1 is a right-angled triangle and so calculate its area.

(b) In triangle T_2, a perpendicular of height h is dropped to the side of length 109.

i. Show that this perpendicular cuts the base into two sections, with lengths

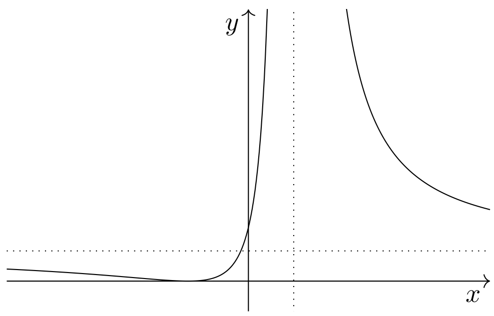
$$\sqrt{11^2 - h^2}, \quad 109 - \sqrt{11^2 - h^2}.$$

ii. Using areas, find the exact value of h.

iii. Hence, determine the value of l.

1163. The graph shown below has the following equation, in which a and b are distinct constants:

$$y = \frac{a^2 + 2ax + x^2}{b^2 + 2bx + x^2}.$$



Give the equations of the two asymptotes.

1164. Two dice are rolled n times, where n is large. Show that a sum of seven on the two dice will occur around twice as often as a sum of nine.

1165. A polynomial graph $y = \mathrm{f}(x)$, of odd degree, has exactly two distinct roots. Prove that, at at least one of these roots, the gradient must be zero.

1166. There are n routes from A to B. Counting a trip and its reverse as distinct, write down the number of different return trips, if the routes out and back

(a) may be the same,

(b) must be different.

1167. Lines L_1, L_2 and L_3 have equations

$$\begin{aligned} L_1 &: y = x, \\ L_2 &: y = -(2+\sqrt{3})x, \\ L_3 &: y = (\sqrt{3}-2)x. \end{aligned}$$

The acute angle between lines L_1 and L_2 is 60°. Show that the acute angle between lines L_1 and L_3 is also 60°.

1168. Solve $(\sqrt{x} + x)^3 = 1$.

1169. Either prove or disprove the following statement: "If two rectangles have the same perimeter as each other and the same area as each other, then they must be congruent."

1170. An aeroplane is climbing slowly, after take-off, at constant velocity. Explain, referring to Newton's laws, how you know that the magnitudes of the following pairs of forces are equal:

(a) the lift experienced by the plane; the weight of the plane,

(b) the weight of the plane, the gravitational force of the plane on the Earth,

(c) the thrust on the plane; the drag on the plane.

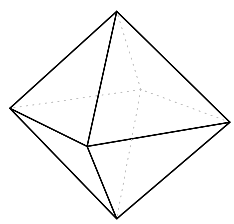
1171. The cubic $y = x^3 - x + 2$ has two stationary points and one point of inflection. Show that these three points are collinear.

1172. State, giving a reason, which of the implications $\Longrightarrow$, $\Longleftarrow$, $\Longleftrightarrow$ links the following statements concerning a real number x:

① $(x-a)(x-b)(x-c) = 0$,

② $(x-a)(x-b) = 0$.

1173. A regular octahedron is shown below.



Two of the vertices are selected at random. Find the probability that they are joined by an edge.

1174. Prove that, if two distinct parabolae of the form $y = \mathrm{f}(x)$ are tangent at a point, then they cannot intersect elsewhere.

1175. A function has instruction

$$\mathrm{f} : x \longmapsto \frac{1}{\sqrt{6 - x - x^2}}.$$

(a) Sketch the curve $y = 6 - x - x^2$.

(b) Hence, or otherwise, show that f is well defined over the domain $[-2, 1]$.

1176. Solve $x - (\sqrt{x} - 1)^3 = 1$.

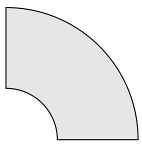
1177. An eccentric individual of mass 60 kg is standing on bathroom scales in a lift which is accelerating upwards at $\frac{1}{3}g$ ms^{-2}. Determine the reading, in kg, on the scales.

1178. The indefinite integral of the function g is a cubic. Show that $y = \mathrm{g}(x)$ has a stationary point.

1179. One of the following statements is true; the other is not. State, with a reason, which is which.

(a) $xyz^{-1} = 0 \implies x = 0$,
(b) $xy^{-1}z^{-1} = 0 \implies x = 0$.

1180. The region shown in the diagram is bordered by arcs of two concentric circles, and by two straight edges of length 1. Its interior angles are all right angles. The perimeter of the shaded region is $P = 4 + \frac{\pi}{2}$.



Show that area of the shaded region is $\frac{1}{4}\pi + 1$.

1181. Explain why one of the following expressions is well-defined and the other is not:

① $\lim_{h \to 0} \dfrac{(x+h)^2 - x^2}{h}$

② $\left.\dfrac{(x+h)^2 - x^2}{h}\right|_{h=0}$

1182. Two vectors **a** and **b** have unit length. Their x and y components, in $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$, are such that $a_1b_1 + a_2b_2 = 0$.

(a) Show that **a** and **b** are perpendicular.
(b) Hence, show that $|\mathbf{a} - \mathbf{b}| = \sqrt{2}$.

1183. State, with a reason, whether the following claims are true in the Newtonian system:

(a) "On an object whose mass is negligible, the resultant force must be negligible."
(b) "On an object whose weight is negligible, the resultant force must be negligible."

1184. Disprove the following statement:

$$\text{If } \mathrm{f}^2(x) \equiv \mathrm{g}^2(x), \text{ then } \mathrm{f}(x) \equiv \mathrm{g}(x).$$

1185. A regular hexagon has two adjacent vertices at the points $(0, 0)$ and $(6, 8)$.

(a) Find the side length.
(b) Show that the area of the hexagon is $150\sqrt{3}$.

1186. A function f is such that $0 \le \mathrm{f}(x) \le 1$ for all x in the domain $\mathbb{R}$. State, with a reason, whether the following are necessarily true, with k as a positive constant,

(a) $k\,\mathrm{f}(x) \in [0, k]$ for all $x \in \mathbb{R}$,
(b) $x \mapsto k\,\mathrm{f}(x)$ has range $[0, k]$ over $\mathbb{R}$.

1187. Differentiate $y = \sqrt{x}$ from first principles.

1188. Write the following in simplified interval notation

(a) $\{x \in \mathbb{R} : |x| < 2\} \cap [1, 3]$,
(b) $\{x \in \mathbb{R} : |x| > 2\} \cap [1, 3]$,
(c) $\{x \in \mathbb{R} : |x| \le 2\} \cap [1, 3]$.

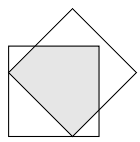
1189. A transformation takes the graph $y = x^2 + 3x + 1$ onto the graph $y = x^2 - 3x + 1$. Describe this transformation as

(a) a reflection,
(b) a translation.

1190. A firework rocket weighing 2 kg blasts off from rest at ground level. A vertical driving force of 31.6 Newtons acts on the rocket for 4.0 seconds, until the fuel is exhausted. From that point on, the rocket is modelled as a projectile. Assume the mass is constant throughout.

(a) Find the speed at which the rocket is travelling when the fuel runs out.
(b) Determine the greatest height achieved by the rocket during the motion.

1191. Two unit squares are placed as depicted below. All acute angles in the diagram are 45°.



Determine the exact area of the shaded region.

1192. The equation $\mathrm{f}(x) = 0$ has exactly one root, at $x = a$. Solve the following equations, giving your answers in terms of a:

(a) $\mathrm{f}(2x) = 0$,
(b) $\mathrm{f}(x-1)\,\mathrm{f}(x+1) = 0$.

1193. Disprove the following statement: "The difference of any two irrational numbers is irrational."

1194. A student tries to solve the equations

$$3x = 13 + 4y,$$
$$6x - 8y = 7.$$

She finds that neither elimination nor substitution produces a solution point (x, y). Explain what the correct interpretation of this fact is.

1195. Show that, if $y = x^2$, then

$$\int_0^k y\,dx + \int_0^{k^2} x\,dy = k^3.$$

1196. The two roots of the quadratic $x^2 + px + q = 0$ differ by 4 and their sum is 2. Find p and q.

1197. In this question, vectors **a** and **b** are non-parallel and non-zero.

(a) Determine the values of p and q to make the following an identity:

$$p(\mathbf{a} + \mathbf{b}) + q(\mathbf{a} - \mathbf{b}) \equiv 4\mathbf{a} + 6\mathbf{b}.$$

(b) Explain how you used the facts that **a** and **b**

i. are non-parallel,

ii. are non-zero.

1198. An AP has first term a and common difference d. Prove that, for such a sequence:

(a) the nth term is given by $u_n = a + (n-1)d$,

(b) the sum of the first n terms is given by

$$S_n = \tfrac{1}{2}n\left(2a + (n-1)d\right).$$

1199. Prove that, if h is a linear function defined over $\mathbb{R}$, then, for any $a \neq b$, the following is constant:

$$\frac{\mathrm{h}(b) - \mathrm{h}(a)}{b - a}.$$

1200. Eliminate a from the following equations, to find a relation of the form $\mathrm{f}(x) = \mathrm{g}(y)$.

$$2ax + y = 4,$$
$$3x - ay = 1.$$

—— End of 12th Hundred ——